MPL–638 **RMSV**

M.Phil. DEGREE EXAMINATION – JUNE 2006.

Mathematics

COMMUTATIVE RINGS

Time : 3 hours Maximum marks : 75

Answer any FIVE questions.

Each question carries 15 marks.

1. (a) Let A be a ring $\neq 0$. Show that the set of prime ideals of A has minimal elements with respect to inclusion.

(b) Let $g : A \to B$ be a ring homomorphism such that $g(s)$ is a unit in B for all $s \in S$. Show that there exists a unique ring homomorphism $h : S^{-1}A \to B$ such that $g = h \circ f$.

2. (a) State and prove the first uniqueness theorem in primary decomposition.

(b) Let K be a field. Show that in the polynomial ring $K[x_1, x_2, x_3, \dots x_n]$ the ideals $pi = (x_1, x_2, x_3 \dots x_i)$ $(1 \le i \le n)$ are prime and all their powers are primary.

3. (a) Let $A \subseteq B$ be integral domains, B integral over A. Prove that B is a field iff A is a field.

(b) Let K be a field and B a finitely generated K-algebra. If B is a field prove that it is a finite algebraic extension of K.

4. (a) M is a Noetherian A-module $\Leftrightarrow$ every sub module of m is finitely generated – Prove the statement.

(b) Let M be an A–module. If every non empty set of finitely generated submodules of M has a maximal element than prove that M is Noetherian.

5. (a) If A is Noetherian, then prove that the polynomial ring $A[x]$ is Noetherian.

(b) If a finitely generated ring K is a field, prove that it is a finite field.

6. (a) Prove that an Artin ring A is a unique finite direct product of Artin local rings.

(b) Let A be an Artin local ring. Prove the following are equivalent.

(i) every ideal in A is principal.

(ii) the maximal ideal m is principal.

7. (a) Let A be a Noetherian domain of dimension 1. Prove that every non-zero ideal a in A can be uniquely expressed as a product of primary ideals whose radicals are all distinct.

(b) Prove that the ring of integers in an algebraic number field K is a Dedekind domain.