6562/KAA **OCTOBER 2007**

Optional — FUNCTIONAL ANALYSIS

(For those who joined in July 2003 and after)

Time : Three hours Maximum : 100 marks

SECTION A — (4 × 10 = 40 marks)

Answer any FOUR questions.

1. If M be a linear subspace of a normed linear space N, and f be a functional defined on M, then prove that f can be extended to a functional f_0 defined on the whole space N such that $\|f_0\| = \|f\|$.

2. State and prove the Uniform Boundedness theorem.

3. If M is closed linear subspace of a Hilbert space H; x is a vector not in M and d is the distance from x to M, then prove that there exists a unique vector y_0 in M such that $\|x - y_0\| = d$.

4. If N_1 and N_2 are normal operators on H with the property that either commutes with the adjacent of the other, then prove that $N_1 + N_2$ and $N_1 N_2$ are normal.

5. If I is a proper closed two-sided ideal in A, then prove that the quotient algebra A/I is a Banach Algebra.

6. If f_1 and f_2 are multiplicative functionals on A with the same null space M, then prove that $f_1 = f_2$.

7. Explain the following :

(a) Conjugate space of X

(b) Dual basis

(c) Completion of the n/s X

(d) Adjoint of F.

8. For a compact operator A on a Banach space X prove that $Z(A-I)$ and $Z(A'-I)$ are equal where A' is the transpose of A.

SECTION B — $(3 \times 20 = 60$ marks)

Answer any THREE questions.

All questions carry equal marks.

9. Let M be a closed linear subspace of a normed linear space N. Show that N/M is also a normed linear space and that if N is a Banach space then so is N/M.

10. If B and B' are Banach space and if T is a continuous linear transformation B onto B', then prove that the image of each open sphere centred on the origin in B contains an open sphere centred on the origin in B'.

11. (a) Show that l_2 is an inner product space.

(b) State and prove the Bessel's inequality.

12. Prove that $r(x) = \lim \|x^n\|^{1/n}$ where $r(x)$ is the spectral radius of an element x in the general Banach algebra A.

13. State and prove the Gelfand–Neumark theorem.

14. Let $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For $y \in L^q([a,b])$. Let $f_y \in (L^p([a,b]))'$ be defined by

$$f_y(x) = \int_a^b x\,y\,dm, x \in L^p([a,b]).$$

Let $F : L^q([a,b]) \to (L^p([a,b]))'$ be given by $F(y) = f_y$, $y \in L^q[[a,b]]$. Then prove that F is a linear isometry of $L^q([a,b])$ into $(L^p([a,b]))'$ and also F is onto if and only if $1 \le p < \infty$.